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CAPTURE SET COMPUTATION OF AN OPTIMALLY GUIDED MISSILE

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Abstract. The paper presents a game theoretical approach for numerically computing the capture set of an optimally guided medium range air-to-air missile against a given target. Realistic point mass models are used, as long flight times prevent simplifications like coplanarity or constant speed target. The capture set is obtained by constructing saddle point trajectories on its boundary, or the barrier, numerically. Instead of solving a game of kind, the trajectories are identified by setting up an auxiliary game of degree. The necessary conditions of the auxiliary game are shown to coincide with those of the game of kind. The game of degree is solved from systematically varied initial states with a decomposition method that does not require setting up or solving the necessary conditions. Examples are calculated for a generic fighter and a missile.

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1 Introduction

A missile is in general designed to be faster and more agile than any aircraft. This kinematical advantage of the missile is, however, only temporary due to a finite and relatively short burn time of its rocket motor. In the coasting phase the kinetic energy of the missile is rapidly dissipated by the aerodynamic drag force. In contrast to a missile, an aircraft can maintain its velocity as long as it has any fuel left. The asymmetry means that a missile does not necessarily reach the aircraft from an arbitrary launch position, but only from within a finite shooting range. The range depends on many factors, such as the performance and initial energy of the missile and target, the guidance law of the missile, the geometry of the shoot and moreover, on the maneuvering of the target. The set of launch positions that are inescapeable for the target is called the 'capture set' (e.g., [1]) or the 'no escape envelope' (e.g. [2]).

An estimate of the capture set is crucial, for example, in assessing the threat related to each opponent in an air combat, and furthermore, in considering actions to be taken. On the other hand, the unit cost of a single missile is usually significant, which calls for minimizing the number of premature shoots.

A worst case estimate for the capture set with given vehicle models can be obtained by assuming that the missile uses a guidance law that produces the largest possible capture set. The situation can then be modeled as a pursuit-evasion game of kind, see Refs. [3, 4], where the missile is identified with the pursuer and the aircraft with the evader. The aim in a game of kind is to identify the set of those initial states from which the optimally behaving pursuer can enforce a capture against any action of the evader. This paper presents a computational method to numerically determine solutions to the game of kind for adversary missile-aircraft encounters with realistic point mass models.

If the missile is assumed to use a known, perhaps nonoptimal, feedback guidance law, the capture set can be found via optimization of the evader's actions. Imado and Miwa [5, 6, 7] have studied maneuvers that lead to a largest possible miss distance against missiles employing either Proportional Navigation (PN) or Augmented Proportional Navigation (APN). Shinar and Guelman [8] present a similarly optimal evasion strategy against a short range PN missile with a simplified model, whereas Ong and Pierson [9] study optimal evasion of a surface-to-air PN missile with a slightly simplified model. Although some of the papers deal with quantitative evasion strategies, the results can be used in assessing the largest effective shooting distance as well.

Nevertheless, a general tendency seems to be towards better guidance schemes, because most existing feedback laws are nonoptimal with excessive target maneuvers [10, 11], long launch distances [12] and large initial deviations from the collision course [1]. If the guidance law of the missile is unknown, pursuit-evasion games provide a possibility to estimate the capture set under the worst case assumption on the guidance law of the missile.

Isaacs [3] provides a unified procedure for solving a pursuit-evasion game of kind by identifying the barrier, a surface that envelopes the initial states leading to a capture. Shinar et al. and Green et al. apply the approach of Isaacs to a simplified short range missile scenario in Refs. [1, 13]. Le Menec and Bernhard [14] use a similar model for barrier computation in an expert system for air combat. To allow solutions in closed form, the models used above assume coplanarity of the players and constant velocity of the evader. Unfortunately, these assumptions lose their validity as the duration of the encounter increases.

For game models that do not allow analytical solutions, a numerical solution scheme is needed. Grimm and Schaeffer [2] approximate the barrier by assuming more realistic vehicle models but a near optimal feedback control for the evader, and optimize the farmost point from which a capture is still possible. Breitner et al. [15, 16] have investigated the determination of the barrier with slightly simplified point mass models in the vertical plane. The approach is based on the numerical solution of the necessary conditions of the saddle point trajectories on the barrier.

In this paper we consider rather realistically modeled flight vehicles maneuvering in three dimensions. Instead of solving the necessary conditions of a game of kind we construct an auxiliary game of degree with the shooting range as the payoff and show that the open-loop representation of a feedback saddle point solution of this game satisfies the necessary conditions of a barrier saddle point trajectory. This is equivalent with the fact that the initial state obtained by solving the auxiliary game of degree lies on the barrier. Points on a submanifold of the barrier, corresponding to partly fixed initial states of the players, are obtained by varying the initial geometry of the encounter and by repeatedly solving the auxiliary game of degree by a numerical decomposition method introduced in Refs. [17] and [18]. The main advantage of this method is that the solution is obtained without explicitly stating or solving the necessary conditions of a saddle point. Instead, two optimal control problems are solved iteratively using discretization and nonlinear programming, until the saddle point has been found. Hence the method offers an easy and rapid treatment of complex game models. To the author's knowledge, three-dimensional capture set computation with present models has not been reported earlier.

The paper is organized as follows. First, the dynamics of the players are introduced. The game of kind is then formulated, the necessary conditions of a barrier trajectory are briefly given, and the barrier submanifolds of interest are described. Next, the auxiliary game of degree is formulated and it is shown that a solution to the auxiliary game satisfies the necessary conditions of a barrier trajectory. The decomposition method for the game of degree is shortly reviewed before numerical examples and concluding remarks.

2 The game dynamics

In the following, the missile is identified with the pursuer P and the aircraft with the evader E. Both P and E maneuver in three dimensions. We make the following simplifying assumptions:

- The thrust and the drag forces, as well as the velocity vectors, are assumed parallel with the reference line of the vehicles.
- The lift force is assumed to be orthogonal to the velocity vector and to point upwards in the frame of reference of either vehicle.
- The inertias of the flight vehicles are assumed negligible.
- The coordinate frame assumes a flat Earth.

Then, the equations of motion are

$$\begin{aligned}
\dot{x}_i &= v_i \cos\gamma_i \cos\chi_i && (1)\\
\dot{y}_i &= v_i \cos\gamma_i \sin\chi_i && (2)\\
\dot{h}_i &= v_i \sin\gamma_i && (3)\\
\dot{\gamma}_i &= \frac{g}{v_i}(n_i \cos\mu_i - \cos\gamma_i) && (4)\\
\dot{\chi}_i &= \frac{g}{v_i}\frac{n_i \sin\mu_i}{\cos\gamma_i}, \quad i = P, E; && (5)\\
\dot{v}_P &= \frac{1}{m_P(t)}[T_P(t) - && (6)\\
& \quad D_P(h_P, v_P, M(h_P, v_P), n_P)] - g\sin\gamma_P && (7)\\
\dot{v}_E &= \frac{1}{m_E}[\eta_E T_{E,max}(h_E, M(h_E, v_E)) - D_E(h_E, v_E, M(h_E, v_E), n_E)] \\
&- g\sin\gamma_E. && (8)
\end{aligned}$$

The state of the game is described by the state vector

$$z = [z'_P, z'_E]' = [x_P, y_P, h_P, \gamma_P, \chi_P, v_P, x_E, y_E, h_E, \gamma_E, \chi_E, v_E]'. \qquad (9)$$

The subscripts P and E refer to the pursuer and evader, respectively, and the prime denotes a transpose. The state variables x_i, y_i, h_i, γ_i, χ_i and v_i, $i = P, E$, are the x and y coordinates, altitudes, flight path angles, heading angles and velocities of the players. The term 'shooting range' that is used throughout the paper, refers to the quantity $[(x_P(0) - x_E(0))^2 + (y_P(0) - y_E(0))^2]^{1/2}$, i.e., the initial distance of the players in the xy-plane.

The flight direction of the vehicles is controlled with the load factors n_P and n_E and the bank angles $\mu_P, \mu_E \in [-\pi, \pi]$. The velocity of E is controlled by the throttle setting $\eta_E \in [0, 1]$ that selects the fraction of the maximal available thrust force $T_{E,max}(h, M(h, v))$, where M is the Mach number. The pursuer's thrust force $T_P(t)$ is in general a fixed function of time that cannot be controlled. The masses of the vehicles are denoted by m_P and m_E and the gravitational acceleration by g.

The drag forces of E and P are assumed to obey a quadratic polar (we suppress the subscripts for brevity),

$$D(h, v, M(h, v), n) = C_{D_0}(M(h, v))Sq(h, v) + n^2 C_{D_I}(M(h, v))\frac{(mg)^2}{Sq(h, v)}, \qquad (10)$$

where $C_{D_0}(\cdot)$ and $C_{D_I}(\cdot)$ denote the zero drag and induced drag coefficients, S the reference wing area and $q(h, v) = 1/2\varrho(h)v^2$ the dynamic pressure. The air density $\varrho(h)$ and the Mach number are computed using the standard ISA atmosphere.

The load factors n_P and n_E cannot be chosen freely. At low velocities, a large load factor requires a large angle of attack which results in loss of lift force and stall. At higher velocities, the magnitudes of the load factors are constrained by the largest accelerations that the flight vehicles and the pilot tolerate. Here, the bounds are approximated by the box constraints

$$n_P \in [0, n_{P,max}] \tag{11}$$
$$n_E \in [0, n_{E,max}]. \tag{12}$$

To summarize, we require that the players' control vectors $u_P(t)$ and $u_E(t)$ satisfy

$$u_P(t) := [n_P(t), \mu_P(t)]' \in S_P := [0, n_{P,max}] \times [-\pi, \pi] \tag{13}$$
$$u_E(t) := [n_E(t), \mu_E(t), \eta_E(t)]' \in S_E := [0, n_{E,max}] \times [-\pi, \pi] \times [0, 1]. \tag{14}$$

In addition, the players have to stay in their flight envelopes. For the evader, the boundaries of interest are the minimum altitude constraint

$$h_E \geq h_{E,min} \tag{15}$$

and the dynamic pressure constraint

$$q(h_E, v_E) \leq q_{E,max}. \tag{16}$$

The pursuer must obey the minimum altitude constraint

$$h_P \geq h_{P,min}. \tag{17}$$

3 Game of kind

Let us assume that the players, obeying (1)-(8) and (13)-(17) have perfect information on the state of the game. In the game of kind, as defined by Isaacs [3], the objective of the evader is to avoid the capture, whereas the pursuer aims at it. A capture occurs when $z(t)$ enters a target set Λ. In this paper, Λ is defined as a set of points where the distance of the players is smaller than a given capture radius d. The boundary $\partial\Lambda$ of Λ is given by the terminal manifold or capture condition

$$l(z(T)) =$$
$$(x_P(T) - x_E(T))^2 + (y_P(T) - y_E(T))^2 + (h_P(T) - h_E(T))^2 - d^2 = 0. \tag{18}$$

The capture condition also implicitly specifies the terminal time T. The solution of a game of kind is the set of all admissible initial conditions $z(0) = z_0$ from which the pursuer can achieve the capture against any admissible control of the evader. If there exists another set of initial states from which the pursuer can not enforce the capture, the two sets (excluding the interior of Λ) are separated by a piecewise continuously differentiable hypermanifold called the barrier. The barrier is a collection of saddle point solution trajectories, also termed as barrier trajectories, that satisfy

$$\max_{u_E} \min_{u_P} \frac{\mathrm{d}}{\mathrm{dt}} l(z(T)) = 0; \tag{19}$$

see [3, 15]. An infinitesimal change of the initial state or a deviation of either player from the optimal saddle point strategy on the barrier immediately results in a capture or escape. Instead of an exhaustive search in the state space, the game of kind can be solved by determining the barrier, or equivalently, the saddle point trajectories forming it.

It should be marked that in the present problem the functional limits of the missile can give rise to additional capture set boundaries. For example, a proximity fuse may require a certain minimum launch distance. These boundaries, however, are not explored here.

Necessary conditions of a barrier trajectory

The necessary conditions, that a barrier trajectory satisfies, consist of the

i) equations of motion

ii) control and state variable inequality constraints

iii) optimal controls of the players that minimaximize the Hamiltonian of the game at each time instant

iv) adjoint differential equations

v) interior point conditions for the transitions from a solution arc to another

vi) jump conditions for the adjoint trajectories at some of the transitions

vii) terminal condition

viii) initial values of the state variables and final conditions for the adjoint variables.

The initial state is selected so that condition (19) holds. The final conditions of the adjoint variables are given as

$$\lambda(T^*) = \alpha \frac{\partial}{\partial z} l(z^*(T^*)), \tag{20}$$

where $\lambda(t)$ is the adjoint vector, α is a positive multiplier, and star denotes a saddle point solution.

For the game at hand, the condition i) is given by (1)-(8), condition ii) by (13)-(17) and condition vii) by (18). The explicit form of conditions iii)-vi) is not needed here, but can be found, e.g., in Ref. [19] and the literature cited therein. The condition (19) results in the following constraint for the final flight path angles, heading angles and the velocities of the players:

$$v_P(T) = v_E(T) \frac{\Delta x \cos\gamma_E(T) \cos\chi_E(T) + \Delta y \cos\gamma_E(T) \sin\chi_E(T) + \Delta h \sin\gamma_E(T)}{\Delta x \cos\gamma_P(T) \cos\chi_P(T) + \Delta y \cos\gamma_P(T) \sin\chi_P(T) + \Delta h \sin\gamma_P(T)}. \tag{21}$$

Here $\Delta x = x_P(T) - x_E(T)$, $\Delta y = y_P(T) - y_E(T)$ and $\Delta h = h_P(T) - h_E(T)$. The final conditions of the adjoint variables yield

$$\lambda_{x_P}(T^*) = -\lambda_{x_E}(T^*) = 2\alpha(x_P^*(T^*) - x_E^*(T^*)) \tag{22}$$
$$\lambda_{y_P}(T^*) = -\lambda_{y_E}(T^*) = 2\alpha(y_P^*(T^*) - y_E^*(T^*)) \tag{23}$$
$$\lambda_{h_P}(T^*) = -\lambda_{h_E}(T^*) = 2\alpha(h_P^*(T^*) - h_E^*(T^*)). \tag{24}$$
$$\tag{25}$$

In addition,

$$\lambda_{v_P}(T^*) = \lambda_{v_E}(T^*) = 0 \tag{26}$$
$$\lambda_{\gamma_P}(T^*) = \lambda_{\gamma_E}(T^*) = 0 \tag{27}$$
$$\lambda_{\chi_P}(T^*) = \lambda_{\chi_E}(T^*) = 0, \tag{28}$$

as the corresponding state variables are free at $t = T^*$.

Barrier submanifolds

In principle, the barrier could be identified by integrating the necessary conditions i)-viii) backwards in time, starting from the transversality conditions (22)-(28) and every point satisfying condition (19) on $\partial\Lambda$ [3]. Nevertheless, the barrier is an 11-dimensional hypermanifold in the 12-dimensional state space, and its construction as such would be a formidable task. Besides, many initial states, like the ones where the pursuer is initially heading away from the evader, hardly bear any practical significance. We therefore concentrate on a submanifold of the barrier defined in the following.

First, since the atmosphere is assumed laterally homogenous, we fix

$$x_E(0) = y_E(0) = 0. \tag{29}$$

For the same reason we can fix

$$y_P(0) = 0, \tag{30}$$

and let $\chi_E(0)$ vary. Consequently, the shooting range is simply given by $x_P(0)$. As we anticipate long solution times, initial transients related to the flight path angle of the evader are considered negligible, and we use

$$\gamma_E(0) = 0. \tag{31}$$

To further decrease the dimension, we fix the initial velocity and altitude of the evader and the initial velocity of the pursuer:

$$v_E(0) = v_{E,0}, h_E(0) = h_{E,0} \tag{32}$$
$$v_P(0) = v_{P,0}; \tag{33}$$

in the computations they will be treated as parameters. Finally, we let the pursuer select its initial flight path angle and heading angle freely,

$$\gamma_P(0) = \gamma_P^*(0) \tag{34}$$
$$\chi_P(0) = \chi_P^*(0). \tag{35}$$

Conventionally, one would assume that at the moment of the launch the target has to be in the visual field of the missile's sensor mechanisms, dictated by the initial direction of the missile. This would give rise to an initial condition of the form

$$k(x_P(0), h_P(0), \gamma_P(0), \chi_P(0)) \leq 0. \tag{36}$$

Nevertheless, as most long range missiles nowadays can download midcourse navigation information from an aircraft that is even not necessarily the launching one, constraining is considered unnecessary. The additional necessary conditions corresponding to (34) - (35) are

$$\lambda_{\gamma_P}(0) = \lambda_{\chi_P}(0) = 0. \tag{37}$$

The resulting submanifold of the barrier lies in the intersection of the hyperplanes (29)-(33) and the manifolds (34)-(35) and is two-dimensional. It specifies the maximal shooting range $x_P^*(0) > 0$ as a function of the initial geometry described by $h_P(0)$ and $\chi_E(0)$. To determine the $x_P^*(0)$ corresponding to $h_P(0) = h_{P,0}$ and $\chi_E(0) = \chi_{E,0}$, one could, for example, solve the multipoint boundary value problem resulting from the necessary conditions i)-viii) numerically, see [15, 16]. In the following, however, we develop an alternative approach that utilizes the decomposition method described in Ref. [17].

4 Auxiliary game of degree

The fact that the total impulse of the missile is finite allows one to set up a game of degree with the shooting range as the payoff. Let us, in addition to (29)-(33), fix also $h_P(0) = h_{P,0}$ and $\chi_E(0) = \chi_{E,0}$. Consider then the game

$$\max_{u_E} \min_{u_P, x_P(0), \gamma_P(0), \chi_P(0)} \quad -\kappa\xi(T) \tag{38}$$

$$\dot{z}_P = f_P(z_P, u_P, t), \qquad z_P(0) = [x_P(0), 0, h_{P,0}, v_{P,0}, \gamma_P(0), \chi_P(0)]' \tag{39}$$

$$\dot{z}_E = f_E(z_E, u_E), \qquad z_E(0) = [0, 0, h_{E,0}, v_{E,0}, 0, \chi_{E,0}]' \tag{40}$$

$$\dot{\xi} = 0, \qquad \xi(0) = x_P(0) \tag{41}$$

$$u_E(t) \in S_E, \qquad u_P(t) \in S_P \tag{42}$$

$$h_E(t) \geq h_{E,min}, \qquad h_P(t) \geq h_{P,min} \tag{43}$$

$$q(h_E(t), v_E(t)) \leq q_{max}, \tag{44}$$

$$l(z(T)) = 0. \tag{45}$$

The payoff is always negative, as the scaling factor κ is positive. The absolute value of $\xi(T)$ equals the shooting range. The state equations (39)-(40) are as in (1)-(8).

We now postulate that a solution that satisfies the necessary conditions of a saddle point of the auxiliary game of degree also satisfies the necessary conditions of a barrier trajectory. Checking this is straightforward for the items i)-vii) on the list in the previous section, assuming that the switching structure of the solution of the game of degree coincides with that of the barrier trajectory associated with the computed initial state. As for item viii) we note the following. The final conditions of the adjoint variables in the game of degree are

$$\mu_{x_P}(\tilde{T}) = -\mu_{x_E}(\tilde{T}) = 2\beta(\tilde{x}_P(\tilde{T}) - \tilde{x}_E(\tilde{T})) \quad (46)$$
$$\mu_{y_P}(\tilde{T}) = -\mu_{y_E}(\tilde{T}) = 2\beta(\tilde{y}_P(\tilde{T}) - \tilde{y}_E(\tilde{T})) \quad (47)$$
$$\mu_{h_P}(\tilde{T}) = -\mu_{h_E}(\tilde{T}) = 2\beta(\tilde{h}_P(\tilde{T}) - \tilde{h}_E(\tilde{T})) \quad (48)$$
$$\mu_{v_P}(\tilde{T}) = \mu_{v_E}(\tilde{T}) = 0 \quad (49)$$
$$\mu_{\gamma_P}(\tilde{T}) = \mu_{\gamma_E}(\tilde{T}) = 0 \quad (50)$$
$$\mu_{\chi_P}(\tilde{T}) = \mu_{\chi_E}(\tilde{T}) = 0 \quad (51)$$
$$\mu_{\xi}(\tilde{T}) = -\kappa, \quad (52)$$

where $\mu(t)$ is the adjoint vector and $(\tilde{\cdot})$ refers to the solution of the game of degree. Conditions (46)-(51) coincide with conditions (22)-(28). Obviously, $\mu_{\gamma_E}(0) = \mu_{\chi_E}(0) = 0$, which yields condition (37). Note that condition (52) is not totally decoupled; using calculus of variations it is rather easy to show that the initial condition (41) implies $\mu_\xi(0) = -\mu_{x_P}(0)$. Since both $\mu_\xi(t)$ and $\mu_{x_P}(t)$ are constant, $\mu_\xi(\tilde{T}) = -\kappa = -\mu_{x_P}(\tilde{T})$. On the other hand, $\mu_{x_P}(\tilde{T})$ should satisfy condition (46). Nevertheless, as long as $\tilde{x}_P(\tilde{T}) > \tilde{x}_E(\tilde{T})$, $\kappa > 0$ can be selected freely without affecting the solution. In the computations of this paper the assumption above holds.

Finally, substituting (46)-(52) to the necessary condition

$$\tilde{H}(\tilde{z}(\tilde{T}), \tilde{u}_P(\tilde{T}), \tilde{u}_E(\tilde{T}), \mu(\tilde{T}), \tilde{\xi}(\tilde{T}), \tilde{T}) \doteq$$
$$\mu'(\tilde{T})[f_P(\tilde{z}_P(\tilde{T}), \tilde{u}_P(\tilde{T}), \tilde{T})', f_E(\tilde{z}_E(\tilde{T}), \tilde{u}_E(\tilde{T}))']' + \mu_\xi(\tilde{T}) \cdot 0 = 0 \quad (53)$$

of the game of degree, and canceling the common factor $\beta > 0$ shows that $\tilde{z}(\tilde{T})$ satisfies condition (21). This concludes that $\tilde{z}(t)$, $t \in [0, \tilde{T}]$ satisfies the necessary conditions of a barrier trajectory. Especially the initial state

$$\tilde{z}_P(0) = [\tilde{x}_P(0), 0, h_{P,0}, v_{P,0}, \tilde{\gamma}_P(0), \tilde{\chi}_P(0)]', z_E(0) = [0, 0, h_{E,0}, v_{E,0}, 0, \chi_{E,0}]' \quad (54)$$

belongs to the barrier. Thus, the maximal shooting range $x_P^*(0)$ corresponding to $h_P(0) = h_{P,0}$ and $\chi_E(0) = \chi_{E,0}$ on the barrier submanifold defined earlier is given by $\tilde{x}_P(0)$. Parts of the submanifold can now be produced by systematically varying $h_P(0)$ and $\chi_E(0)$ and solving the game of degree (38)-(45) repeatedly. Different submanifolds are obtained by varying $v_{E,0}$, $h_{E,0}$ and $v_{P,0}$ in (32)-(33). In this way, for example, the effect of the initial velocity of the evader on the shooting range can be assessed.

5 Solving the game of degree

The approach presented above is computationally intensive. A reliable method to solve the auxiliary game of degree is provided in Ref. [17]. In the method the game is decomposed into two subproblems that are solved iteratively until the saddle point is reached. Since the subproblems are standard optimal control problems, they can be solved efficiently using discretization and nonlinear programming. The complex and problem dependent necessary conditions are not explicitly involved in the solution process, but the solution can be shown to satisfy them, see Ref. [17]. We next give a short description of the method when applied to the game above. For clarity, we eliminate the dummy state variable ξ and consider directly the minimaximization of $x_P(0)$.

If we assume that min and max operators commute, the maxmin solution will coincide with the saddle point solution. To obtain the maxmin solution, consider first the minimization problem

$$\min_{u_P, x_P(0), \gamma_P(0), \chi_P(0), T} \quad -x_P(0) \tag{55}$$

$$\dot{z}_P = f_P(z_P, u_P, t), \qquad z_P(0) = [x_P(0), 0, h_{P,0}, v_{P,0}, \gamma_P(0), \chi_P(0)] \tag{56}$$

$$u_P(t) \in S_P, \qquad h_P(t) \geq h_{P,min} \tag{57}$$

$$l(z_P(T), z_E^0(T)) = 0, \tag{58}$$

where $z_E^0(t)$, $t \geq 0$ is a fixed feasible trajectory of the evader, and $l(\cdot)$ is defined as in (18). Let the solution trajectory be $\bar{z}_P(t)$ and the final time $\bar{T}$ and denote the capture point $z_E^0(\bar{T})$ by $\bar{e}$. Now, $\bar{z}_P(t)$ is also an optimal trajectory for the problem where the evader's trajectory above is replaced by the fixed point $\bar{e}$ and the final time is fixed to $\bar{T}$. In the neighborhood of $(\bar{e}, \bar{T})$ define

$$\begin{aligned} V(e,T) \;=\; & \min_{u_P, x_P(0), \gamma_P(0), \chi_P(0)} \{-x_P(0) \mid \dot{z}_P = f_P(z_P, u_P, t), \\ & t \in [0,T],\ z_P(0) = [x_P(0), 0, h_{P,0}, v_{P,0}, \gamma_P(0), \chi_P(0)],\ u_P(t) \in S_P, \\ & h_P(t) \geq h_{P,min},\ l(z_P(T), e) = 0\}. \end{aligned} \tag{59}$$

The maxmin problem (38)-(45) now reduces to maximizing $V(e,T)$ subject to the evader's constraints. This problem is difficult to solve since $V(\cdot)$ cannot be expressed analytically. We therefore fix the final time to $\bar{T}$, linearize $V(e, \bar{T})$ in the neighborhood of $\bar{e}$ and solve the free final state problem

$$\max_{u_E} \quad \frac{\partial V}{\partial e}(\bar{e}, \bar{T})(z_E(\bar{T}) - \bar{e}) \tag{60}$$

$$\dot{z}_E = f_E(z_E, u_E),\ t \in [0,T],\ z_E(0) = z_{E0} \tag{61}$$

$$u_E(t) \in S_E, \qquad h_E(t) \geq h_{E,min},\ q(h_E(t), v_E(t)) \leq q_{E,max}. \tag{62}$$

Above, gradient is a row vector. Basic sensitivity results applied to V (see Ref. [17]) imply that the gradient of V with respect to e at $(\bar{e}, \bar{T})$ is given by the analytical expression

$$\frac{\partial}{\partial e} V(\bar{e}, \bar{T}) = \frac{\partial}{\partial e}\xi(\bar{T}) + \bar{\alpha}\frac{\partial}{\partial e} l(\bar{z}_P(\bar{T}), \bar{e}) = \bar{\alpha}\frac{\partial}{\partial e} l(\bar{z}_P(\bar{T}), \bar{e}), \tag{63}$$

where $\bar{\alpha}$ is the Lagrange multiplier associated with the capture condition (58) in the solution of (55)-(58).

Denote the solution trajectory of (60)-(62) by $z_E^1(t)$, $t \in [0, \bar{T}]$. Also denote $T^1 = \bar{T}$. Finally, insert the extended solution

$$z_E^1(t) = \begin{cases} z_E^1(t), & t \leq T^1 \\ z_E^1(T^1) + \dot{z}_E^1(T^1)(t - T^1), & t > T^1, \end{cases} \tag{64}$$

back to (55)-(58). Repeat the procedure to obtain $z_E^2(t)$, and continue until the value of the payoff or $z_E^k(t)$ no more changes. In Ref. [17] it is shown that the limit solution of the iteration satisfies the necessary conditions of an open-loop representation of a feedback saddle point. Convergence of the method is discussed in Ref. [18]. The method is especially suitable for computing the capture set, since with suitably small initial state discretization intervals of $h_{P,0}$ and $\chi_{E,0}$, the solution of the previous problem usually lies in the convergence domain of the next problem.

6 Numerical Examples

In the following, we compute parts of the submanifolds of the barrier corresponding to two initial altitudes and velocities of the evader, and one initial velocity of the pursuer. Although the study is limited, many informative results already appear. We use a generic medium range air-to-air missile model. The thrust force of the rocket motor is given by

$$T_P(t) = \begin{cases} T_B, & 0 \le t \le 3 \text{ s} \\ T_S, & 3 < t \le 15 \text{ s} \\ 0, & t > 15 \text{ s}. \end{cases} \tag{65}$$

Consequently, the mass of the missile first decreases piecewise linearly and remains then constant. The mass of the evader is assumed constant. The drag coefficients of both vehicles are approximated by rational polynomials on the basis of realistic tabular data. The tabular data describing $T_{E,max}(h_E, M(h_E, v_E))$ is approximated by a two-dimensional polynomial. We set $n_{E,max} = 7$, $n_{P,max} = 20$, and $q_{E,max} =$ 80 [kPa]. The evader's data used in the numerical examples represents a high-performance fighter aircraft. For details, the reader is urged to the literature cited in Ref. [19].

The subproblems of the method presented in the previous section are discretized using collocation with cubic polynomials, see Refs. [20, 21, 22]. In this method, the solution time interval is discretized using a suitable discretization grid. The state trajectories between the gridpoints are approximated with piecewisely defined cubic polynomials. The controls are approximated with piecewise linear continuous functions. The approximating cubics are required to be continuous and smooth in their first derivative. In addition, in the middle of each interval, the slopes of the cubics are required to coincide with the values of the state equations. For a review of different discretization schemes, see Ref. [23].

In the pursuer's problem, the phases of the missile are discretized separately, and solution arcs are joined by appropriate continuity conditions. The evader's problem is discretized equidistantly. Both problems employ 40 discretization points. The resulting nonlinear programming problems are solved using NPSOL [24] library subroutine, which is a versatile implementation of Sequential Quadratic Programming, see e.g., [25]. The use of Lagrangian Merit function substantially improves the convergence of the method from an almost arbitrary infeasible initial point.

In each example, the pursuer's initial velocity $v_{P,0}$ is fixed to 300 m/s. The pursuer's initial altitude is varied from $h_{P,0} = h_{P,min}$ m to $h_{P,0} = 9,000$ m in $1,000$ m intervals. The initial heading angle of the evader is varied from $\chi_{E,0} = 0°$ to $\chi_{E,0} = 180°$ in 18° intervals. Here 0° means a head on shoot and 180° a tail shoot. Note that the situation is symmetrical for $\chi_{E,0} > 180°$. With the selected initial state discretization, $10 \times 11 = 110$ game problems are solved for each combination of $h_{E,0}$, $v_{E,0}$ and $h_{P,0}$. We trade precision with computing time and do not require ultimate accuracy in the solution of individual optimization problems. The decomposition method is terminated once the relative change in the shooting distance becomes less than 0.5%.

Example 1

First, we study a case where the evader initially flies at $h_{E,0} = 3,000$ m with the initial velocity $v_{E,0} = 400$ m/s in level flight. A plot of the maximal shooting ranges corresponding to the pursuer's different initial altitudes and directions of shoot is presented in Figure 1. The effect of the atmosphere is visible in the lower part of the manifold. When the pursuer starts from a low altitude, it stays in the denser part of the atmosphere during the flight. As a result, the maximal shooting range is relatively small. However, when the initial altitude of the pursuer increases, it can reach thinner atmosphere, and the maximal shooting range grows almost linearly with the initial total energy of the pursuer.

When the pursuer is initially in a low altitude, the maximal shooting range clearly depends on the direction of the shoot. In higher initial altitudes, however, the dependence is alleviated. An intuitive explanation is as follows. In every case, and especially in a head on shoot, the evader's optimal strategy is to turn away from the missile regardless of the pursuer's initial altitude, even though the turn itself does not increase the distance to the pursuer. The duration of the turn is roughly constant, whereas the terminal time increases with the initial altitude of the pursuer. Thus, with evader's heading angles near 0° and with low initial altitudes of the pursuer, the evader uses most of the available flight time in turning, whereas in solutions corresponding to high initial altitudes of the pursuer, the fraction of the total time the evader uses for turning is smaller.

In addition to turning away from the pursuer, the evader also ascends to avoid the dynamic pressure constraint that becomes active towards the end of the encounter. The pursuer's strategy is to avoid atmospheric drag by climbing relatively much. Consequently, the optimal initial flight path angle of the pursuer is often more than 60°. Some comparisons indicate that constraining the initial flight path angle and heading angle such that the target is required to be within a cone of visibility with an opening angle of 35° shows a decrease of the magnitude of 10% in the maximal shooting range. The difference is mainly due to the momentary but large loadfactor needed to steer the pursuer upwards and laterally towards the final position of the evader. Thus, the significance of the free initial direction of the pursuer seems to be noteworthy.

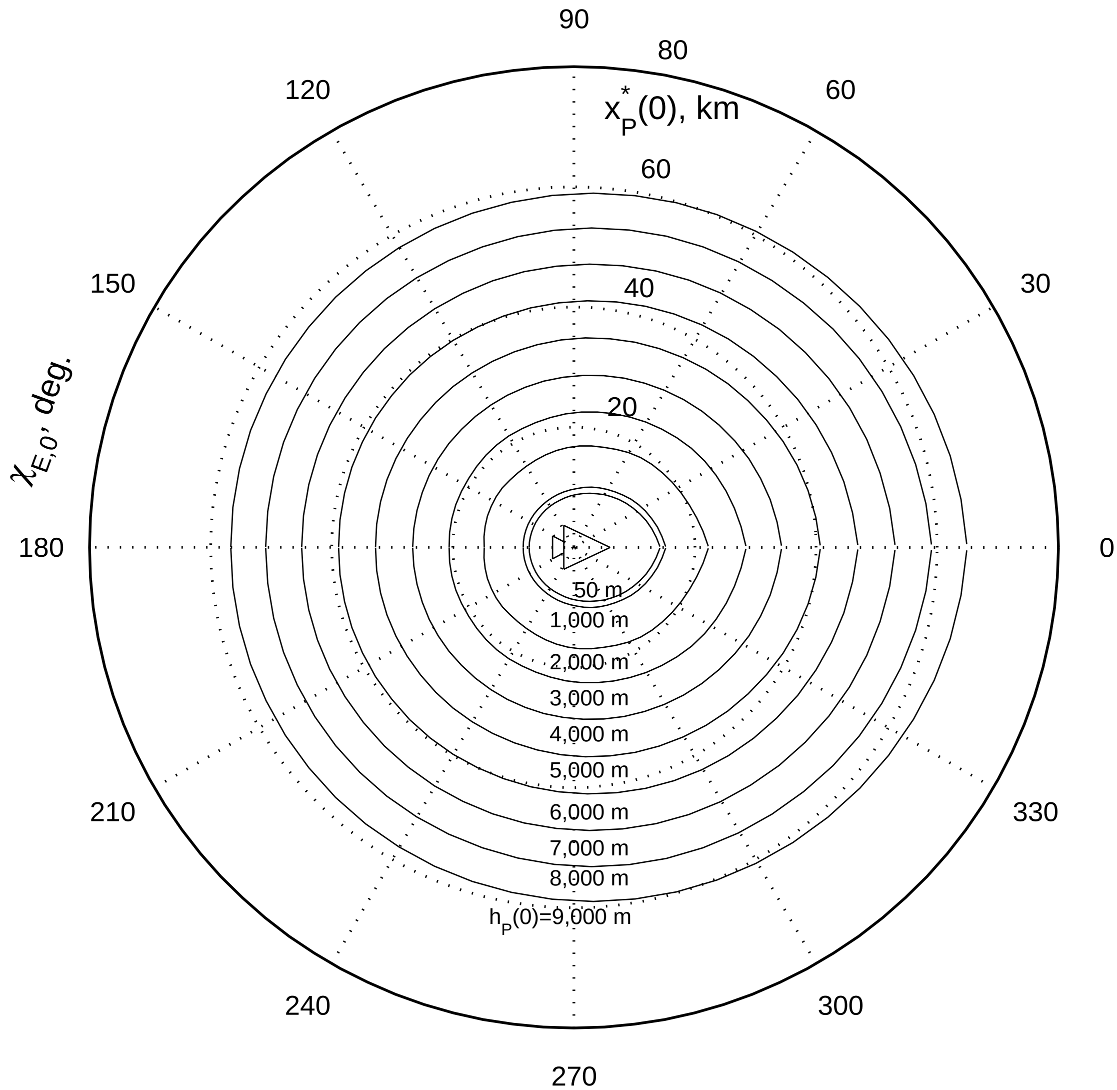

Figure 1: The shooting range as a function of the pursuer's initial altitude and the direction of the shoot in example 1. The evader lies in the origin and flies to the right.

Example 2

In example 2, the initial velocity of the evader is decreased to 250 m/s. The maximal shooting range is presented as a function of the pursuer's initial altitude and the direction of shoot in Figure 2. It is interesting to note that with higher initial altitudes of the pursuer, the maximal shooting range grows less than 10% on the average when compared to the previous example. One would expect that the lower initial energy of the evader would be more advantageous for the pursuer. The thrust excess, i.e., the difference of the thrust force and the drag force of the evader is, however, larger with lower velocities, and acceleration to almost the same velocity as with the higher initial velocity takes only a fraction of the terminal time. The velocity difference thus becomes almost compensated in the first moments of the flight. In the lower parts of the barrier submanifold the difference is larger, since the

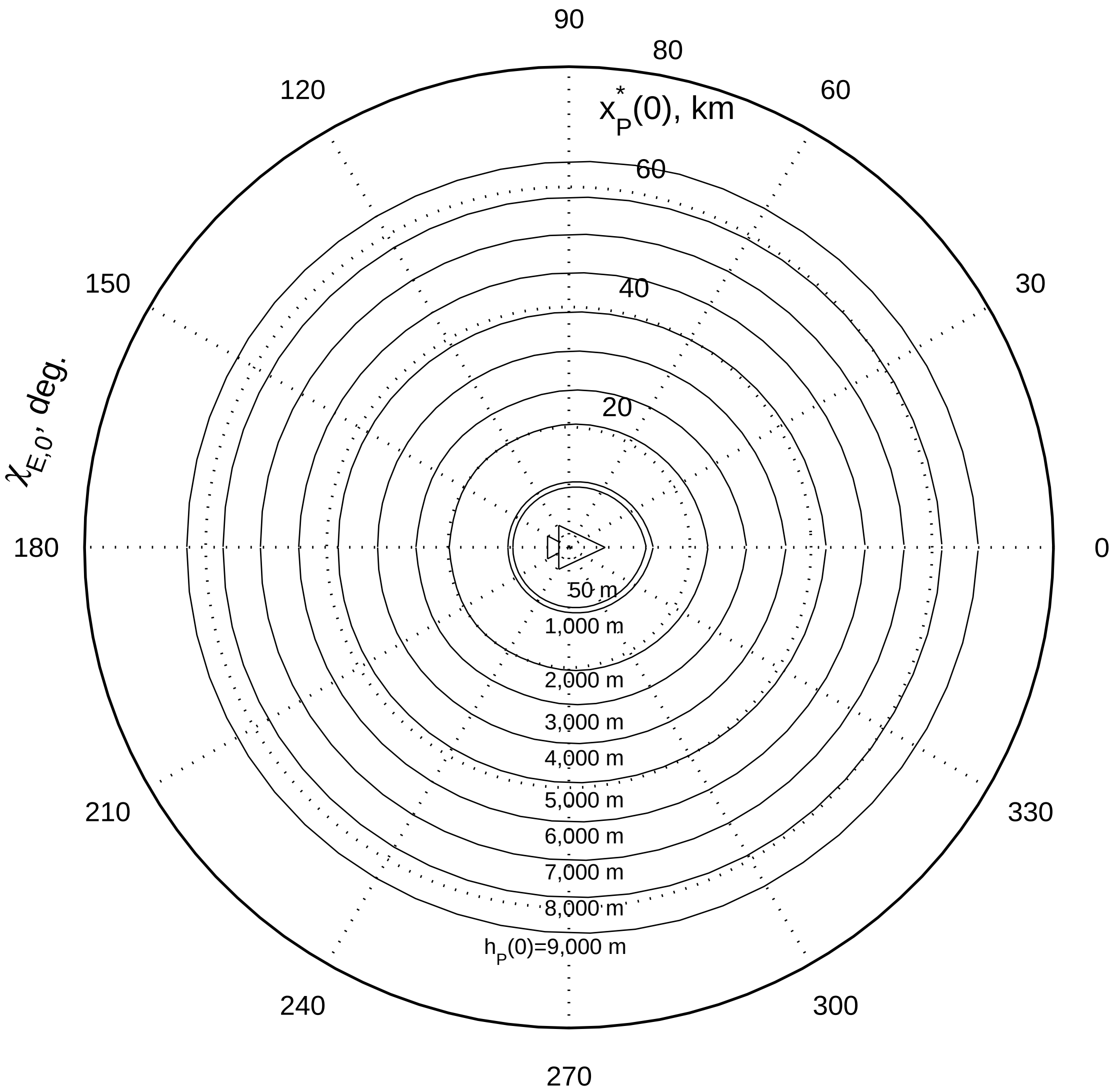

Figure 2: The shooting range as a function of the pursuer's initial altitude and the direction of the shoot in example 2.

acceleration takes more than half of the total flight time. On the other hand, the largest possible shooting range for a head on shoot is smaller than in the previous example. The turning performance of the evader is better at the lower initial velocity.

Example 3

Finally, we compute the maximal shooting ranges for an evader at $h_{E,0} = 6,000$ m with the initial velocity $v_{E,0} = 400$ m/s. The maximal shooting ranges are shown as functions of the pursuer's initial altitude and the direction of shoot in Figure 3. The higher initial altitude of the evader extenuates the maximal shooting ranges. In the pursuer's initial altitudes of $2000 - 3000$ m, the change is approximately 40% compared to the first example. In other altitudes it is of the order of $10 - 15\%$.

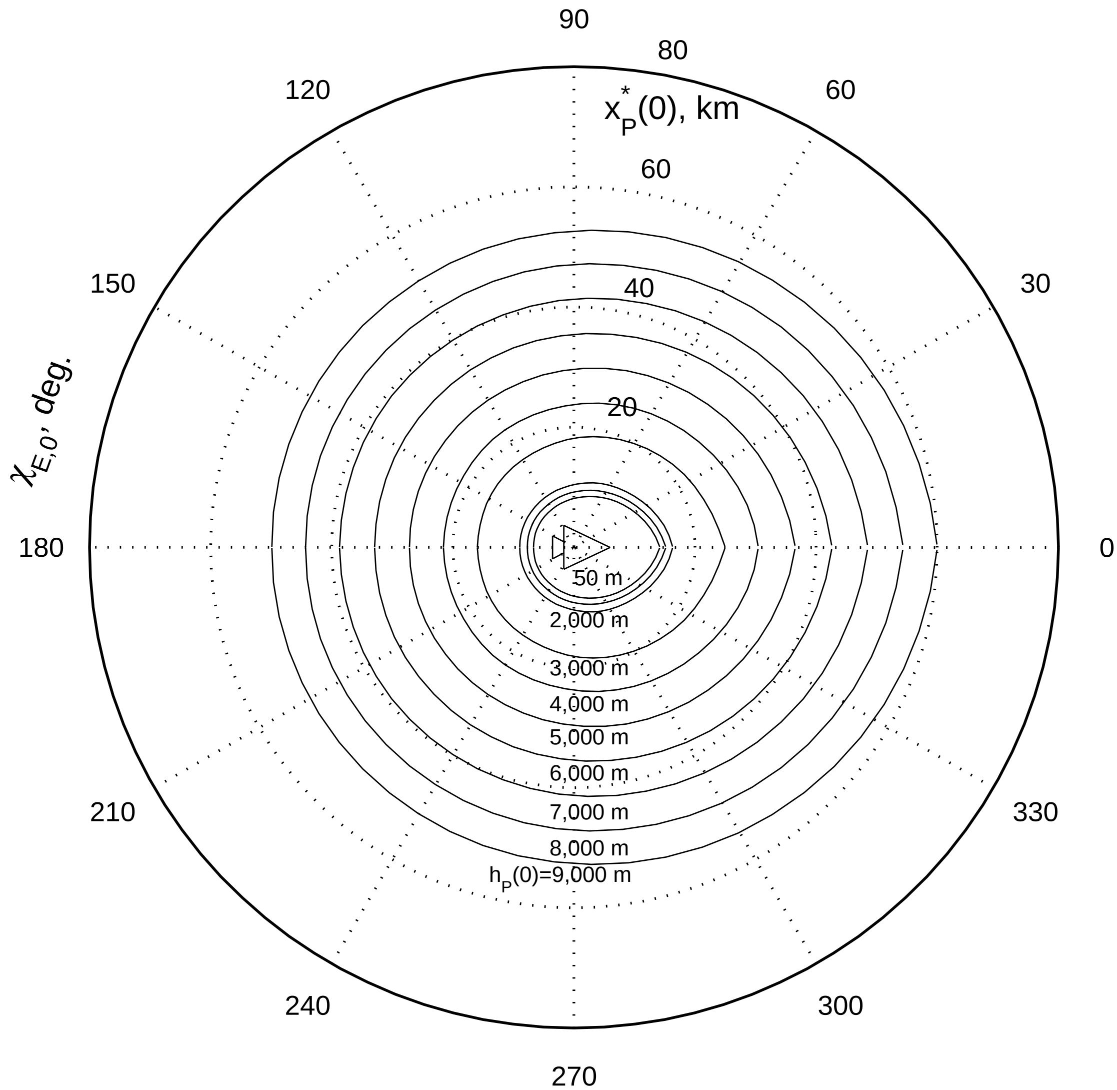

Figure 3: The shooting range as a function of the pursuer's initial altitude and the direction of the shoot in example 3.

7 Conclusions

We have presented a method to compute the capture set of a missile with given vehicle models under the assumption that both the missile and the aircraft behave in the best possible way. Instead of solving the necessary conditions of the corresponding game of kind, an equivalent game of degree is set up. A submanifold of the barrier in three dimensions is constructed by solving the game of degree repeatedly with suitably discretized initial states using a decomposition method presented earlier. The necessary conditions of the game need not be solved explicitly. The use of discretization and nonlinear programming in solving the subproblems of the decomposition offers a tradeoff possibility between accuracy and solution time. A coarse solution with only few discretization points can be obtained fast for inspection, and, if desired, can be improved by adding of reallocating the discretization points. More efficient computation could be achieved with optimization algorithms designed especially for sparse problems, and possibly parallelization.

The presented demonstrations show, as expected, that the maximal shooting range depends on the altitude difference of the players and, through the properties of the atmosphere, also on the absolute initial altitude of the pursuer. The shape of the lower part of the no escape envelope depends largely on the direction of the shoot, but in the upper part this dependence is diluted.

The computations indicate that the shooting range is rather insensitive to small changes in the evader's trajectory. On one hand, this means that a successful implementation of the evasive maneuvers is not crucially affected by small variations. On the other hand, since the maximization in the presented decomposition method is essentially a first-order approach, it may suffer from convergence difficulties near such a flat optimum. Nonetheless, as pointed out in Ref. [18], the maximization problem can be solved using any method of nonlinear programming, including second-order algorithms. Another possibility to obtain more accurate results is to initiate an indirect solution method with the obtained solution. The initial estimate of the adjoint variables can be calculated using the Lagrange multipliers of the converged subproblems.

The approach provides a way to assess the technical performance of the vehicles in the worst possible case, which is important especially if the guidance law of the missile is unknown. It should be noted, however, that in this paper the information pattern is unmodeled and assumed perfect. Usually it is the aircraft that has problems in receiving information on the missile. For example, detecting a missile launch is a challenging task. If a missile can hide itself, the largest possible shooting range thus becomes larger.

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